

Overpartitions Identities Involving Gaps and Weights

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Abstract.

In this paper, we find an identity which connects the overpartition function and the function of Rogers–Ramanujan–Gordon type overpartitions by considering the weights and gaps. This identity can be seen as an analogue of the weighted identity of Alladi.

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1 Introduction

In 1997, Alladi [1] began a systematic study of weighted partition identities. By attaching weights to the gaps between parts of partitions, he obtained interesting new identities connecting the famous partition functions of Euler, Gauss, Lebesgue, Rogers–Ramanujan and others. In this paper we apply the definition of the successive Durfee squares for overpartitions given by Corteel and Mallet [9] to give an identity connecting the Rogers–Ramanujan–Gordon type overpartitions and the overpartitions without any restrictions, which can be seen as an analogue of the weighted identity of Alladi.

To illustrate the weighted partition identities he obtained, Alladi proposed the following general problem: Given a set S of partitions, let $P_S(n)$ denote the number of partitions π of n with $\pi \in S$. Suppose $S \subset T$. The problem is to determine weights $w_S(\pi) \geq 1$ in a natural way such that for all n

$$P_T(n) = \sum_{\sigma(\pi)=n, \pi \in S} w_S(\pi). \quad (1.1)$$

(Here and in what follows, $\sigma(\pi)$ is the sum of the parts of π .)

Naturally, we want to know whether we could get some new identities, if the sets S and T in above problem are replaced by sets of overpartitions. We focus on the situation that S denotes the set of the Rogers–Ramanujan–Gordon type overpartitions and T denotes the set of the overpartitions without any restrictions.

Recall that a partition λ of a positive integer n is a finite non-increasing sequence of positive integers $(\lambda_1, \lambda_2, \dots, \lambda_l)$ such that $\sum_{i=1}^l \lambda_i = n$. Let $p(n)$ denote the number of partitions of n .

An overpartition is a partition for which the first occurrence of a part may be overlined. For example, $(\overline{7}, 7, 6, \overline{5}, 2, \overline{1})$ is an overpartition of 28. The number of overpartitions of n is denoted by the function $\overline{p}(n)$.

For a partition λ , we use the same symbol to denote its Ferrers graph. The conjugate of the Ferrers graph of λ is denoted by λ^* .

Given two partitions λ_1 and λ_2 , by $\lambda_1 + \lambda_2$ we mean the partition whose parts are obtained by adding the number of nodes in the corresponding rows of the Ferrers graphs of λ_1 and λ_2 . If λ_1 is an overpartition, then the parts of $\lambda_1 + \lambda_2$ in the corresponding rows of the Ferrers graph of overlined parts of λ_1 remain overlined.

Here and in the rest of this paper, we employ the customary q-series notations

$$(a)_\infty = (a; q)_\infty = \sum_{i=0}^{\infty} (1 - aq^i),$$

$$(a)_n = (a; q)_n = \frac{(a)_\infty}{(aq^n)_\infty}$$

and

$$(a_1, \dots, a_k; q)_\infty = (a_1; q)_\infty \cdots (a_k; q)_\infty.$$

By these notations, we recall the generating functions of the partitions and overpartitions

$$\sum_n p(n)q^n = \frac{1}{(q; q)_\infty}, \quad (1.2)$$

$$\sum_n \bar{p}(n)q^n = \frac{(-q)_\infty}{(q)_\infty}. \quad (1.3)$$

By a combinatorial proof, Corteel and Lovejoy [8] gave another form of the generating function of overpartitions

$$\sum_n \bar{p}(n)q^n = \frac{(-q)_\infty}{(q)_\infty} = \sum_{n=0}^{\infty} \frac{(-1; q)_n q^{n(n+1)/2}}{(q; q)_n^2}, \quad (1.4)$$

which will be employed to demonstrate the relation between the overpartitions without any restrictions and the Rogers-Ramanujan-Gordon type overpartitions.

The theorem of the general Rogers-Ramanujan-Gordon type overpartitions was given by Chen, Sang and Shi [7] which can be seen as an analogue of Gordon's [10] combinatorial generalization of the Rogers-Ramanujan identities. The definition of the Rogers-Ramanujan-Gordon type overpartition is subtracted from this theorem which is stated as follows.

Definition 1.1 For $k \geq 2$ and $k \geq i \geq 1$, we define the Rogers-Ramanujan-Gordon type overpartition λ to be the overpartition such that $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ with part 1 occurs as a non-overlined part at most $i - 1$ times, and $\lambda_j - \lambda_{j+k-1} \geq 1$ if λ_j is overlined and $\lambda_j - \lambda_{j+k-1} \geq 2$ otherwise. Let $D_{k,i}(n)$ denote the number of such overpartitions λ of n .

Chen, Sang and Shi [7] also derived the following generating function form of $D_{k,i}(n)$ as well. For $k \geq 2$ and $k \geq i \geq 1$, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} D_{k,i}(n)q^n \\ &= \sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1(N_1+1)/2 + N_2^2 + \cdots + N_{k-1}^2 + N_{i+1} + \cdots + N_{k-1}} (-q)_{N_1-1} (1 + q^{N_i})}{(q)_{N_1-N_2} \cdots (q)_{N_{k-2}-N_{k-1}} (q)_{N_{k-1}}}. \end{aligned} \quad (1.5)$$

Setting $i = k$ in (1.5), we get the generating function of $D_{k,k}(n)$

$$\begin{aligned} \sum_{n=0}^{\infty} D_{k,k}(n)q^n \\ = \sum_{N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_1(N_1+1)/2 + N_2^2 + \dots + N_{k-1}^2} (-1)^{N_1}}{(q)_{N_1-N_2} \cdots (q)_{N_{k-2}-N_{k-1}} (q)_{N_{k-1}}}. \end{aligned} \quad (1.6)$$

Applying the definition of successive Durfee squares for overpartitions, we could give new combinatorial explanations for (1.4) and (1.6). Recall that in 1979, Andrews [3] introduced the idea of Durfee dissection of an integer partition to interpret the generalized Rogers-Ramanujan identity combinatorially. To extend this work to overpartitions, Corteel and Mallet in [9] introduced the generalized Durfee square for overpartitions.

In Section 2, we shall introduce the definition of successive Durfee squares for overpartitions, and give new combinatorial explanations for (1.4) and (1.6). Then in Section 3, we will apply these new explanations to give the definition of $w_S(\pi)$ where $P_T(n) = \bar{p}(n)$ and $P_S(n) = D_{k,k}(n)$ in (1.1).

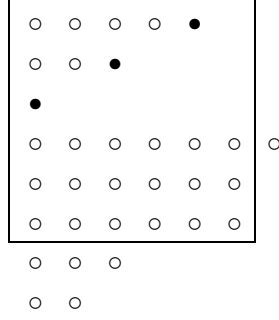
2 Successive Durfee Squares

In this section, we shall introduce the definition of successive Durfee squares for overpartitions, and give new combinatorial explanations for (1.4) and (1.6).

Firstly, we will introduce the definition of generalized Durfee square.

Definition 2.1 (See [9].) *We say that the generalized Durfee square of an overpartition λ has size N if N is the largest integer such that the number of overlined parts plus the number of non-overlined parts greater than or equal to N is greater than or equal to N . Let $D(\alpha)$ denote the size of generalized Durfee square of α .*

For example, let $\alpha = (7, 6, 6, \bar{5}, \bar{3}, 3, 2, \bar{1})$. We rewrite it as $\alpha = (\bar{5}, \bar{3}, \bar{1}, 7, 6, 6, 3, 2)$. And we could get that $D(\alpha) = 6$. Therefore, α has a generalized Durfee square of size 6. We present its Ferrers graph as follows.



Inspired by the Algorithm Z [4], Corteel and Mallet in [9] also gave the generating function for overpartitions with generalized Durfee square of size N where the exponent of q counts the weight and the exponent of a counts the number of overlined parts as follows

$$\frac{a^N q^{\frac{(N+1)N}{2}} (-1/a)_N}{(q)_N (q)_N}. \quad (2.1)$$

Following this idea, we can use the definition of generalized Durfee square to explain the common factor both in (1.4) and (1.6)

$$\sum_{n=0}^{\infty} \frac{(-1; q)_n q^{(n+1)n/2}}{(q; q)_n}. \quad (2.2)$$

Theorem 2.2 *Let α denote the overpartition such that the number of parts of α equals the size of generalized Durfee square of α . And $g(n)$ denotes the number of overpartitions α of n . Then the generating function of $g(n)$ is as follows*

$$\sum_{n=0}^{\infty} g(n) q^n = \sum_{N=0}^{\infty} \frac{(-1; q)_N q^{(N+1)N/2}}{(q; q)_N}. \quad (2.3)$$

Proof. Let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_N)$ denote a partition with N distinct parts and $\delta = (\delta_1, \delta_2, \dots, \delta_k)$ denote a partition with distinct parts and $0 \leq \delta_i \leq N - 1$ for $1 \leq i \leq k$. Then we can see that the γ with N parts generates the factor $\frac{q^{(N+1)N/2}}{(q; q)_N}$ and δ generates the factor $(-1; q)_N$. We aim to show that there is a bijection between the set of partition pairs (γ, δ) and the set of overpartitions α , where the number of parts of α is N , the same as the size of its generalized Durfee square.

First, given a partition pair (γ, δ) , we will construct an overpartition α . Before the construction, note the fact that $\gamma_i \geq N - i + 1$, which means $\gamma_i + (i - 1) \geq N$, since γ is a partition with N distinct parts. Now we could begin to construct the overpartition α . We overline all the parts of γ . Then

from 1 to N , if δ has a part $\delta_j = i - 1$, add $i - 1$ to γ_i and remove the overline of γ_i . Finally, we write the overlined parts of γ before the non-overlined parts to get an overpartition α . It is easy to see that there are N parts in α and each non-overlined part in α is no less than N . Then it follows that α is an overpartition with N parts and generalized Durfee square of size N .

For example, if $N = 5$, $\gamma = (7, 6, 5, 2, 1)$ and $\delta = (4, 3, 0)$, then we have

$$\begin{aligned}
(7 + 6 + 5 + 2 + 1, 4 + 3 + 0) &\implies (\overline{7} + \overline{6} + \overline{5} + \overline{2} + \overline{1}, 4 + 3 + 0) \\
&\implies (7 + \overline{6} + \overline{5} + \overline{2} + \overline{1}, 4 + 3) \\
&\implies (7 + \overline{6} + \overline{5} + 5 + \overline{1}, 4) \\
&\implies (7 + \overline{6} + \overline{5} + 5 + 5) \\
&\implies (\overline{6} + \overline{5} + 7 + 5 + 5).
\end{aligned}$$

Clearly, $\alpha = (\overline{6}, \overline{5}, 7, 5, 5)$ have five parts and the size of generalized Durfee square is 5.

Now we present the inverse map. For an overpartition $\alpha = (\alpha_1, \dots, \alpha_l, \alpha_{l+1}, \dots, \alpha_N)$, where α_i is overlined for $1 \leq i \leq l$ and α_j is non-overlined for $l + 1 \leq j \leq N$, and $\alpha_N \geq N$, we construct (γ, δ) as follows.

- (i) Begin with $\gamma = (\alpha_1, \dots, \alpha_l)$ and $\delta = \emptyset$. Remove the overline of α_i for $1 \leq i \leq l$.
- (ii) For j from 1 to $N - l$, choose the smallest $m \geq 0$ such that $\alpha_{l+j} - m > \gamma_{m+1}$, put $\alpha_{l+j} - m$ into γ as a new part, and put m into δ as a new part. If $m + 1 > l(\gamma)$, where $l(\gamma)$ denotes the number of parts of γ , take $\gamma_{m+1} = 0$.

It can be checked that γ is a partition with N distinct parts and δ is a partition with distinct parts between 0 and $N - 1$. And it is indeed the inverse map.

For example, if $N = 5$ and $\alpha = (\overline{6}, \overline{5}, 7, 5, 5)$, then we have

$$\begin{aligned}
(\overline{6} + \overline{5} + 7 + 5 + 5) &\implies (6 + 5, 7 + 5 + 5) \\
&\implies (7 + 6 + 5, 5 + 5 + 0) \\
&\implies (7 + 6 + 5 + 2, 5 + 3 + 0) \\
&\implies (7 + 6 + 5 + 2 + 1, 4 + 3 + 0).
\end{aligned}$$

This completes the proof. ■

Now we proceed to the definition of successive Durfee squares for overpartitions. To get the successive Durfee squares, we first determine the generalized Durfee square of the overpartition, and then the second Durfee square of the smaller partition below the generalized Durfee square, and so on.

For example, let $\alpha = (8, \overline{7}, 6, 6, \overline{5}, 5, 5, 3, \overline{1})$. Then its successive Durfee square dissection is shown as follows.

To construct the map, we have to consider the gaps between parts of the generalized Durfee square, which force us to rearrange the parts in decreasing order. Meanwhile, we must make sure that the other successive Durfee squares do not change. With no loss, we can add N_1 to each overlined part of the generalized Durfee square and rearrange the parts in decreasing order to get a new overpartition, where N_1 is the size of generalized Durfee square. And we can show that after the transformation we could minus N_1 from each overlined part.

Let λ be an overpartition with no restriction and β be an overpartition with at most $k - 1$ successive Durfee squares.

Now we give a surjection $\phi: \lambda \rightarrow \beta$. Let k_λ denote the number of successive Durfee squares of λ .

(i) If $k_\lambda \leq k - 1$, let $\phi(\lambda) = \lambda$.

(ii) If $k_\lambda \geq k$, add N_1 to each overlined part of the generalized Durfee square of λ and rearrange the parts in decreasing order to get a new overpartition λ' , where N_1 is the size of generalized Durfee square of λ . Clearly, the number of successive Durfee squares of λ' is k_λ and the size of the first successive Durfee square of λ' is N_1 . Denote the partition below the $(k - 1)$ -th successive Durfee square of λ' by λ'_b , and denote the partition above λ'_b by λ'_a . Then we minus N_1 from each overlined part of $\lambda'_a + (\lambda'_b)^*$, and get $\phi(\lambda)$ immediately.

It is easy to see that the map ϕ is a surjection. Then we give the inverse map ϕ^{-1} .

For an overpartition β with at most $k - 1$ successive Durfee squares, the sizes of successive Durfee squares are $N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0$. Let k_β denote the number of successive Durfee squares of β .

(i) If $k_\beta \leq k - 2$, let $\phi^{-1}(\beta) = \beta$.

(ii) If $k_\beta = k - 1$, we shall map β to several overpartitions with the number of successive Durfee squares $\geq k - 1$. Generally speaking, we will remove some nodes from the partition to the right of the first successive Durfee square, and place them below the bottom to construct new parts.

step 1. Add N_1 to each overlined part of β and rearrange the parts in decreasing order to get a new overpartition β' . Clearly, the first successive Durfee square of β' consists of the non-overlined parts of the generalized Durfee square of β and the overlined parts of β with each plus N_1 . Then each part in the first successive Durfee square of β' is greater than or equal to N_1 .

Step 2. For i from 1 to N_{k-1} , if β'_{i+1} is overlined and $\beta'_i - \beta'_{i+1} \geq 2$, then we can subtract one from each $\beta'_1 \geq \beta'_2 \geq \dots \geq \beta'_i$, and put a part i under

the $(k - 1)$ -th successive Durfee square of β' as a new part. Then we get a new overpartition with the number of Durfee squares $\geq k$. We can see that there are $\beta'_i - \beta'_{i+1}$ ways of the subtraction. That is, subtracting 0, 1, \dots , or $\beta'_i - \beta'_{i+1} - 1$ from the first i parts of β' .

If β'_{i+1} is non-overlined, and $\beta'_i - \gamma'_{i+1} \geq 1$ then we can subtract one from each $\beta'_1 \geq \beta'_2 \geq \dots \geq \beta'_i$, then we put a part i under the $(k - 1)$ -th Durfee square of β' as a new part. Then we can get an new overpartition with the number of durfee squares $\geq k$. We can see there are $\beta'_i - \beta'_{i+1} + 1$ ways of the subtraction. That is, subtracting 0, 1, \dots , or $\beta'_i - \beta'_{i+1}$ parts from the first i parts of β' .

Step 3. Subtract N_1 from each overlined part and rearrange it.

Then we define the weight $w(\delta)$ to be that

$$w(\delta) = \begin{cases} \prod_{i=1}^{N_{k-1}} (\beta'_i - \beta'_{i+1} + 1 - \varepsilon(\gamma'_{i+1})), & \text{if } \beta \text{ has exactly } k - 1 \text{ Durfee squares,} \\ 1, & \text{if } \beta \text{ has less than } k - 1 \text{ Durfee squares.} \end{cases} \quad (3.1)$$

where

$$\varepsilon(\beta'_{i+1}) = \begin{cases} 1, & \text{if } \beta'_{i+1} \text{ is overlined,} \\ 0, & \text{if } \beta'_{i+1} \text{ is non-overlined.} \end{cases} \quad (3.2)$$

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